Inextensional Free Vibrations of Circular Cylindrical Shells

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Asymptotic solutions are obtained for the eigenvalue problems of the inextensional free vibrations of circular cylindrical shells, considering all 45 possible combinations of the boundary conditions, characterizing the simply supported, the clamped, and the free ends. In addition to the well-known Rayleigh and Love types of inextensional vibrations for shells with the free-free ends, a type represented by a linear combination of those classical ones is found in cases where one end is free and the other is supported in such a manner that it can move freely in the axial direction. The existence of the three types of inextensional mode is proved by an experiment, and the mode shapes are visualized by holographic interferometry.

I. Introduction

THE present paper is concerned with an apparently well-established eigenvalue problem of the inextensional free vibrations of a thin elastic circular cylindrical shell, which assumes that the midsurface of the shell permits no stretching. Rayleigh[1] was the first to find a valid solution to this problem. In *The Theory of Sound*, published in 1877, he presented a closed-form solution for the natural frequencies, assuming that the shell was indefinitely long and that the mode shape was uniform along the generator. Rayleigh's work was criticized by Love[2] in that the assumed mode of the inextensional vibrations led to expressions of the displacements that could not satisfy the boundary conditions at the free ends. But he soon realized that the extensional strain was confined to so narrow a region near the free end that its effect in altering the total amount of the potential energy was negligible and that most of the shell remained practically inextensional. In *A Treatise on the Mathematical Theory of Elasticity*,[3] Love proved that a finite shell that is free at both ends can vibrate without midsurface stretching, not only in the cylindrical mode shape as assumed by Rayleigh, but also in a mode shape that varies linearly along the generator and is antisymmetric with respect to the cross section bisecting the axis of the cylinder. Love's solution for the natural frequencies exhibits dependence on the length of the cylinder.

These developments appear to have established the solutions for the eigenvalue problem. There has been no significant theoretical development since, and state of the art remains essentially the same as it was nearly 100 years ago. It is well accepted nowadays that the inextensional vibrations occur either when the shell has the free-free ends or when it is infinitely long. The mode shape of the inextentional vibrations is to be either of the Rayleigh type or of the Love type; see, e.g., Ref. 4. The natural frequencies of the inextensional vibrations detected in some experiments agree with the theoretical predictions; see, e.g., Ref. 5. None of these experiments, however, seem to have identified the types of the mode shape.

The present paper shows that there exists yet another type of inextensional vibration that occurs when one end is free and the other is supported in such a manner that it can move freely in the axial direction. The existence of such a type of inextensional vibration is proved by an experiment, and its mode shape is visualized by holographic interferometry.

II. Basic Equations

Let us consider a circular cylindrical shell of radius R, thickness h, and length $2L$. The shell is assumed to be made of an elastic material with Young's modulus E, Poisson's ratio ν, and mass per unit volume ρ. The coordinates x and θ are set on the midsurface of the shell such that x measures the distance along the generator from the center of the shell and θ the circumferential angular extent. Time is denoted by t. The displacement components in the axial, circumferential, and lateral directions are denoted by u_x, u_θ, and w_z, respectively, w_z being positive for outward normal to the midsurface. Budiansky's[6] equations for small perturbations of stressed shells are specialized for the free vibrations of the ciruclar cylindrical shell. They are

Kinematic relations:

$$e_x = \frac{\partial u_x}{\partial x}, \qquad e_\theta = \left(\frac{\partial u_\theta}{\partial \theta} + w_z\right)\Big/ R$$
$$e_{x\theta} = \left(\frac{R\partial u_\theta}{\partial x} + \frac{\partial u_x}{\partial \theta}\right)\Big/ 2R$$
$$\kappa_x = -\frac{\partial^2 w_z}{\partial x^2}, \qquad \kappa_\theta = -\left(\frac{\partial^2 w_z}{\partial \theta^2} - \frac{\partial u_\theta}{\partial \theta}\right)\Big/ R^2$$
$$\kappa_{x\theta} = -\left(\frac{4R\partial^2 w_z}{\partial x \partial \theta} + \frac{\partial u_x}{\partial \theta} - \frac{3R\partial u_\theta}{\partial x}\right)\Big/ 4R^2 \tag{1}$$

Constitutive equations:

$$N_x = K(e_x + \nu e_\theta), \qquad N_\theta = K(e_\theta + \nu e_x), \qquad N_{x\theta} = (1-\nu)Ke_{x\theta}$$
$$M_x = D(\kappa_x + \nu\kappa_\theta), \qquad M_\theta = D(\kappa_\theta + \nu\kappa_x), \qquad M_{x\theta} = (1-\nu)D\kappa_{x\theta} \tag{2}$$

where

$$K = Eh/(1-\nu^2), \qquad D = Eh^3/12(1-\nu^2) \tag{3}$$

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Equations of motion:

$$\frac{\partial N_x}{\partial x}+\left(\frac{\partial N_{x\theta}/\partial\theta}{R}\right)-\left(\frac{\partial M_{x\theta}/\partial\theta}{2R^2}\right)-\frac{\rho h\partial^2 u_x}{\partial t^2}=0 \quad (4a)$$

$$\left(\frac{\partial N_\theta/\partial\theta}{R}\right)+\frac{\partial N_{x\theta}}{\partial x}+\left(\frac{\partial M_\theta/\partial\theta}{R^2}\right)+(3/2R)\frac{\partial M_{x\theta}}{\partial x}-\frac{\rho h\partial^2 u_\theta}{\partial t^2}=0 \quad (4b)$$

$$\frac{\partial^2 M_x}{\partial x^2}+(2/R)\frac{\partial^2 M_{x\theta}}{\partial x\partial\theta}+\left(\frac{\partial^2 M_\theta/\partial\theta^2}{R^2}\right)-\frac{N_\theta}{R}-\frac{\rho h\partial^2 w_z}{\partial t^2}=0 \quad (4c)$$

Boundary conditions at $x=L$ and $-L$:

$$w_z=0 \ \text{or} \ Q_x=0, \qquad \frac{\partial w_z}{\partial x}=0 \ \text{or} \ M_x=0$$

$$u_x=0 \ \text{or} \ N_x=0, \qquad u_\theta=0 \ \text{or} \ S_{x\theta}=0 \quad (5)$$

Here, Q_x and $S_{x\theta}$ are the normal and tangential components of the equivalent edge shear, respectively, which are defined by

$$Q_x=\frac{\partial M_x}{\partial x}+(2/R)\frac{\partial M_{x\theta}}{\partial\theta}, \qquad S_{x\theta}=N_{x\theta}+(3/2R)M_{x\theta} \quad (6)$$

The basic equations are expressed in nondimensional form by introducing nondimensional quantities and operators defined by

$$u=u_x/R, \quad v=u_\theta/R, \quad w=w_z/R, \quad y=x/R, \quad T=t/\mu$$

$$N=N_x/K, \quad M=RM_x/D, \quad Q=Q_x/K, \quad S=S_{x\theta}/K$$

$$(\ )'=\frac{\partial(\ )}{\partial y}, \quad (\ )^{\cdot}=\frac{\partial(\ )}{\partial\theta}, \quad (\ )^{*}=\frac{\partial(\ )}{\partial T}$$

$$\nabla^2(\ )=(\ )''+(\ )^{\cdot\cdot} \quad (7)$$

where

$$\mu^2=\rho hR^2/K \quad (8)$$

III. Fundamental Assumptions and Governing Equations

The equations of motion, Eqs. (4), are expressed in terms of the displacements with the aid of Eqs. (1) and (2) to form a system of three simultaneous differential equations for u, v, and w. From the first two, [Eqs. (4a) and (4b)], u and v can be separated. They are then used to express the right-hand members of the constitutive equations, Eqs. (2), in terms of w. The differential equations thus obtained relate N, M, Q, and S to w, which will be referred to here as supplemental equations. The remaining equation of motion, Eq. (4c), is expressed only in terms of w, eliminating u and v with the aid of Eqs. (4a) and (4b). As a result, an eighth-order differential equation for w is obtained, which will be referred to as the free vibration equation. The complete expressions of the supplemental and the free vibration equations are too complicated to present in the space available. It may be sufficient for our purpose to state that the coefficients of these equations are given in polynomials in a geometric parameter δ defined by

$$\delta=h^2/12R^2 \quad (9)$$

From the fundamental assumptions of the thin-shell theory, we may assume that

$$\delta\ll 1 \quad (10)$$

The coefficients of the governing equations can be simplified, therefore, by neglecting terms of order of magnitude δ in their polynomial expressions.

The fundamental solution of the free vibration equation may be written in the form

$$w=\exp(\lambda y)\cos n\theta\sin\omega T \quad (11)$$

where n is the circumferential wave number, ω is a frequency parameter, and λ is the eigenvalue that determines the modal characteristics along the generator.

Let it be assumed that

$$\delta|\lambda^2|\ll 1, \qquad \delta n^2\ll 1, \qquad \omega^2\ll 1 \quad (12)$$

The first two of these assumptions are equivalent to assuming that the wavelength of the deformation or vibrations is much greater than the thickness. The third is a valid assumption for highly flexural vibrations. It follows that

$$\delta|(\ )''|\ll|(\ )|, \qquad \delta|(\ )^{\cdot\cdot}|\ll|(\ )|, \qquad |(\ )^{**}|\ll|(\ )| \quad (13)$$

Accordingly, the governing equations can be simplified further to yield the free vibration equation:

$$\nabla^8 w+8w^{\prime\prime\cdot\cdot\cdot\cdot}+2w^{\cdot\cdot\cdot\cdot\cdot\cdot}+(1-\nu^2)w^{\prime\prime\prime\prime}/\delta+4w^{\prime\prime\cdot\cdot}+w^{\cdot\cdot\cdot\cdot}$$
$$+\{\nabla^4 w-(3+2\nu)w''-w^{\cdot\cdot}\}^{**}/\delta=0 \quad (14)$$

and the supplemental equations:

$$\nabla^4 u=-\nu w'''+w^{\prime\cdot\cdot} \quad (15a)$$

$$\nabla^4 v=-(2+\nu)w^{\prime\prime\cdot}-w^{\cdot\cdot\cdot} \quad (15b)$$

$$\nabla^4 N=(1-\nu^2)w^{\prime\prime\cdot\cdot}+\delta\nu(w^{\cdot\cdot\cdot\cdot\cdot\cdot}+w^{\cdot\cdot\cdot\cdot})-\nu w^{\cdot\cdot **} \quad (15c)$$

$$\nabla^4 M=-\nabla^4(w''+\nu w^{\cdot\cdot})-\nu(2+\nu)w^{\prime\prime\cdot\cdot}-\nu w^{\cdot\cdot\cdot\cdot} \quad (15d)$$

$$\nabla^4 Q=-\nabla^4\{w'''+(2-\nu)w^{\prime\cdot\cdot}\}-3w^{\prime\prime\prime\cdot\cdot}-(2-\nu)w^{\prime\cdot\cdot\cdot\cdot} \quad (15e)$$

$$\nabla^4 S=-(1-\nu^2)w^{\prime\prime\prime\cdot}-\delta(2-\nu)w^{\prime\cdot\cdot\cdot\cdot\cdot}-(2-\nu)\delta w^{\prime\cdot\cdot\cdot}$$
$$+(1+\nu)w^{\prime\cdot **} \quad (15f)$$

Details of the derivation of Eqs. (14) and (15) are found in a previous report[7] by the first author.

IV. Eigenvalues

Substitution of w from Eq. (11) into Eq. (14) yields

$$\lambda^8+A_3\lambda^6+A_2\lambda^4+A_1\lambda^2+A_0=0 \quad (16)$$

where

$$A_3=-4n^2, \qquad A_2=(1-\nu^2)/\delta+6n^4$$
$$A_1=-4n^2(n^2-1)^2+(2n^2+3+2\nu)\omega^2/\delta$$
$$A_0=n^4(n^2-1)^2-n^2(n^2+1)\omega^2/\delta \quad (17)$$

In the present paper, we shall only be concerned with the case where ω is equal to Rayleigh's solution ω_0:

$$\omega_0^2=\delta n^2(n^2-1)^2/(n^2+1) \quad (18)$$

Then, from the last of Eqs. (17),

$$A_0=0 \quad (19)$$

Equation (16) now becomes

$$\lambda^2(\lambda^6 + A_3\lambda^4 + A_2\lambda^2 + A_1) = 0 \tag{20}$$

It can be shown that the roots of Eq. (20) take the form of

$$\lambda_1, \lambda_2 = 0, \qquad \lambda_3, \lambda_4 = \pm n\xi_1$$
$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm n(\xi_2 \pm i\eta_2) \tag{21}$$

where ξ_1, ξ_2, and η_2 are positive real numbers and $i = \sqrt{-1}$. From the root and coefficient relations of Eq. (20), we have

$$\xi_2^2 + 2(\xi_2^2 - \eta_2^2) = 4$$
$$2\xi_1^2(\xi_2^2 - \eta_2^2) + (\xi_2^2 + \eta_2^2)^2 = \Delta^{-2}$$
$$\xi_1^2(\xi_2^2 + \eta_2^2)^2 = (n^2-1)^2(2n^2+1-2\nu)/n^4(n^2+1) \tag{22}$$

where

$$\Delta = n^2/2k^2 = \delta^{1/2}n^2/(1-\nu^2)^{1/2} \tag{23}$$

Approximate solutions for ξ_1, ξ_2, and η_2 satisfying Eqs. (22) are now sought. To this end, let it be assumed that

$$\Delta \ll 1 \tag{24}$$

Then, ξ_1^2, ξ_2^2, and η_2^2 may be expressed in asymptotic series in Δ such as

$$\xi_1^2 = \Delta^2(\xi_{10} + \Delta^2\xi_{11} + \Delta^4\xi_{12} + \cdots) \tag{25a}$$
$$\xi_2^2 = \Delta^{-1}(\xi_{20} + \Delta\xi_{21} + \Delta^2\xi_{22} + \cdots) \tag{25b}$$
$$\eta_2^2 = \Delta^{-1}(\eta_{20} + \Delta\eta_{21} + \Delta^2\eta_{22} + \cdots) \tag{25c}$$

where ξ_{ij} are real and of order of magnitude unity.

The left-hand members of Eqs. (22) are substituted from Eqs. (25), and terms of like powers of Δ on both sides are equated. The result is

$$\xi_{10} = (n^2-1)^2(2n^2+1-2\nu)/n^4(n^2+1) \tag{26a}$$
$$\xi_{20} = \eta_{20} = 1/2 \tag{26b}$$

If terms of order of magnitude Δ are neglected in the series of Eqs. (25), ξ_1, ξ_2, and η_2 become

$$\xi_1 = \Delta\xi_0 + O(\Delta^2), \qquad \xi_2 = \eta_2 = \Delta^{-1/2}/\sqrt{2} + O(\Delta) \tag{27}$$

where

$$\xi_0 = \sqrt{\xi_{10}} = O(1) \tag{28}$$

It follows then that

$$\lambda_1, \lambda_2 = 0, \qquad \lambda_3, \lambda_4 = \pm\Delta n\xi_0 + O(\Delta^2)$$
$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm(1 \pm i)\Delta^{-1/2}n/\sqrt{2} + O(\Delta) \tag{29}$$

It should be noted that the fundamental solutions corresponding to λ_i $(i = 1, 2, 3, 4)$ represent the global solutions which vary gradually over the entire surface of the cylinder, whereas those corresponding to λ_j $(j = 5, 6, 7, 8,)$ represent the edge-zone solutions which decay rapidly as the distance from the ends increases. The general solution for w can now be written as

$$w = \left\{W_1 + W_2y + \sum_{i=3}^{4} W_i \exp(\lambda_i y) + \sum_{j=5}^{8} W_j \exp(\lambda_j y)\right\} \times \cos n\theta \sin\omega T \tag{30}$$

where W_m $(m = 1, 2, 3, \ldots, 8)$ are arbitrary constants.

Let it be assumed that the shell is of moderate length, such that

$$\eta_0 = 1/n\ell = O(1) \tag{31}$$

Then, since

$$\lambda_3 y, \lambda_4 y = \pm\Delta(\xi_0/\eta_0)(y/\ell) = O(\Delta) \tag{32}$$

the exponential functions in the global solutions can be expanded into power series in y/ℓ, and the general solution may be represented by truncating the series by the first four terms. Thus, one has

$$w = \left\{W_1 + W_2y + W_3y^2 + W_4y^3 + \sum_{j=5}^{8} W_j \exp(\lambda_j y)\right\} \times \cos n\theta \sin\omega T \tag{33}$$

It should be noted here that W_3y^2 and W_4y^3 are of order of magnitude Δ^2 and are negligible in comparison with W_1 and W_2y. They will be retained, however, until the final expressions of the boundary conditions are derived.

V. Boundary Conditions

Boundary conditions to be considered in the present paper are those of the simply supported, the clamped, and the free ends. They are defined and designated as

Simply supported ends:

$$S1(w = M = u = v = 0) \tag{34a}$$
$$S2(w = M = u = S = 0) \tag{34b}$$
$$S3(w = M = N = v = 0) \tag{34c}$$
$$S4(w = M = N = S = 0) \tag{34d}$$

Clamped ends:

$$C1(w = w' = u = v = 0) \tag{34e}$$
$$C2(w = w' = u = S = 0) \tag{34f}$$
$$C3(w = w' = N = v = 0) \tag{34g}$$
$$C4(w = w' = N = S = 0) \tag{34h}$$

Free ends:

$$FR(Q = M = N = S = 0) \tag{34i}$$

Each of the homogeneous equations constituting the boundary conditions can be expressed in terms of the unknown coefficients W_m. Those expressions for the equations $w = 0$ and $w' = 0$ are obtained readily from Eq. (33). Those for $u = 0$, $v = 0$, $Q = 0$, $M = 0$, $N = 0$, and $S = 0$ are obtained from the supplemental equations, Eqs. (15): the right-hand members of Eqs. (15) are substituted from Eq. (33), and the particular solutions are sought for those quantities on the left-hand side. The results are set equal to zero at $y = \ell$ and $-\ell$. Here, only those equations for $y = \ell$ are presented. Those for $y = -\ell$ are readily obtained by replacing ℓ with $-\ell$.

$$w = 0: \quad W_1 + \ell W_2 + \ell^2 W_3 + \ell^3 W_4 + \mathbf{1E}(\ell)\mathbf{W} = 0 \tag{35a}$$

$$w' = 0: \quad \Delta^{1/2}B_1(\ell W_2 + 2\ell^2 W_3 + 3\ell^3 W_4) + \mathbf{eE}(\ell)\mathbf{W} = 0 \tag{35b}$$

$$u = 0: \quad \Delta^{-1/2}U_1[\ell W_2 + 2\ell^2 W_3 + (3 + U_2)\ell^3 W_4] - \mathbf{e}^3\mathbf{E}(\ell)\mathbf{W} = 0 \tag{35c}$$

$$v=0: \quad \Delta^{-1}V_1[W_1+\ell W_2+(1+V_2)\ell^2W_3+(1+3V_2)\times\ell^3W_4]+e^2E(\ell)W=0 \tag{35d}$$

$$Q=0: \quad \Delta^{3/2}Q_1[\ell W_2+2\ell^2W_3+(3+Q_2)\ell^3W_4]-e^3E(\ell)W=0 \tag{35e}$$

$$M=0: \quad \Delta M_1[W_1+\ell W_2+(1+M_2)\ell^2W_3+(1+3M_2)\times\ell^3W_4]-e^2E(\ell)W=0 \tag{35f}$$

$$N=0: \quad \Delta N_1[W_1+\ell W_2+(1+\Delta^{-2}N_2)\ell^2W_3+(1+3\Delta^{-2}N_2)\ell^3W_4]-e^2E(\ell)W=0 \tag{35g}$$

$$S=0: \quad \Delta^{3/2}S_1[\ell W_2+2\ell^2W_3+(3+\Delta^{-2}S_2)\ell^3W_4]-e^3E(\ell)W=0 \tag{35h}$$

where

$$W=\{W_5, W_6, W_7, W_8\}^T;\ T \text{ for transpose}$$

$$e^m=2^{-m/2}(1+i)^m\{1,\ (-i)^m,\ (-1)^m,\ (i)^m\};\ m=1,2,3,\ldots$$

$$E(y)=\begin{bmatrix}\exp(\lambda_5y), & 0, & 0, & 0\\ & \exp(\lambda_6y), & 0, & 0\\ & & \exp(\lambda_7y), & 0\\ \text{(sym)} & & & \exp(\lambda_8y)\end{bmatrix} \tag{36}$$

Expressions of B_1, U_1, U_2, V_1, V_2, Q_1, Q_2, M_1, M_2, N_1, N_2, S_1, and S_2 are obtained explicitly in terms of v, η_0, and n^2. It is sufficient for our purpose to state that they are real and of order of magnitude unity.

The reason why the terms ℓ^2W_3 and ℓ^3W_4 have been retained is obvious from Eqs. (35g) and (35h), in which they become of the same order of magnitude as W_1 and ℓW_2. This gives rise to the question of whether the exact expressions should have been used for the supplemental equations relating N and S with w, instead of the first approximation forms as given in Eqs. (15). The result, however, is that the use of the exact expressions of the supplemental equations leads to N and S identical in form with Eqs. (35g) and (35h) except for the expressions of the coefficients N_1, N_2, S_1, and S_2.

The boundary conditions defined by Eqs. (34) are now given by systems of four linear homogeneous equations for W_m $(m=1,2,\ldots,8)$. They are solved for W_i $(i=1,2,3,4)$ in terms of W_j $(j=5,6,7,8)$. Terms of order of magnitude Δ^2 are then neglected in the global solution parts. At this stage ℓ^2W_3 and ℓ^3W_4 are neglected in comparison with W_1 and ℓW_2. Also neglected are terms of order of magnitude Δ in the edge-zone solution part. As a result, the boundary conditions for $y=\ell$ become

$$S1: \quad W_1=-\ell W_2=-\Delta^{1/2}(1/U_1)e^3E(\ell)W,\quad \mathbf{1}E(\ell)W=0,\ e^2E(\ell)W=0 \tag{37a}$$

$$S2: \quad W_1=\ell W_2=0,\ \mathbf{1}E(\ell)W=0,\ e^2E(\ell)W=0 \tag{37b}$$

$$S3: \quad W_1+\ell W_2=0,\ \mathbf{1}E(\ell)W=0,\ e^2E(\ell)W=0 \tag{37c}$$

$$S4: \quad W_1=-\ell W_2=\Delta^{-2}(S_2/S_1)\ell^3W_4-\Delta^{-3/2}\times(1/S_1)e^3E(\ell)W,\ e^2E(\ell)W=0 \tag{37d}$$

$$C1: \quad W_1=\ell W_2=0,\ \mathbf{1}E(\ell)W=0,\ eE(\ell)W=0 \tag{37e}$$

$$C2: \quad W_1=-\mathbf{1}E(\ell)W,\quad \ell W_2=0,\quad eE(\ell)W=0,\quad e^3E(\ell)W=0 \tag{37f}$$

$$C3: \quad W_1=-\ell W_2=\Delta^{-1/2}(1/B_1)eE(\ell)W,\ \mathbf{1}E(\ell)W=0 \tag{37g}$$

$$C4: \quad W_1=\Delta^{-1/2}(1/B_1)eE(\ell)W-\mathbf{1}E(\ell)W,\quad \ell W_2=-\Delta^{-1/2}(1/B_1)eE(\ell)W,\quad e^3E(\ell)W=0 \tag{37h}$$

$$FR: \quad W_1=-\Delta^{-3/2}(1/Q_1)e^3E(\ell)W+\Delta^{-1}(1/M_1)e^2E(\ell)W,\quad \ell W_2=\Delta^{-3/2}(1/Q_1)e^3E(\ell)W \tag{37i}$$

It should be noted here that terms of order of magnitude $\Delta^{3/2}$ have been neglected in the global solution parts in $C3$ and $C4$. In addition, $\lambda_5=-\lambda_7$ and $\lambda_6=-\lambda_8$ and $\exp(-\lambda_5\ell)$ and $\exp(-\lambda_6\ell)$ are negligible in comparison with $\exp(\lambda_5\ell)$ and $\exp(\lambda_6\ell)$.

VI. Nontrivial Solutions

There are 45 possible combinations of the boundary conditions between the ends $y=\ell$ and $-\ell$. For each combination, nontrivial solutions for W_m are sought with the aid of Eqs. (36) and (37).

It can easily be shown that only trivial solutions exist if the combinations contain $S1$, $S2$, $C1$, or $C2$, or if they consist of $S3$, $S4$, $C3$, and $C4$ only. Nontrivial solutions are found when both ends are free (*FR-FR*), and when one end is free and the other is supported in such a manner that it can move freely in the axial direction (*FR-S*3, *FR-S*4, *FR-C*3, and *FR-C*4).

FR-FR

The boundary conditions are given by Eq. (37i) for $y=\ell$ and the corresponding ones for $y=-\ell$:

$$W_1=-\ell W_2+\Delta^{-1}(i/M_1)\{\exp(\lambda_5\ell)W_5-\exp(\lambda_6\ell)W_6\} \tag{38a}$$

$$\ell W_2=-\Delta^{-3/2}\{(1-i)/\sqrt{2}Q_1\}\{\exp(\lambda_5\ell)W_5+i\exp(\lambda_6\ell)W_6\} \tag{38b}$$

$$W_1=\ell W_2+\Delta^{-1}(i/M_1)\{\exp(\lambda_5\ell)W_7-\exp(\lambda_6\ell)W_8\} \tag{38c}$$

$$\ell W_2=-\Delta^{-3/2}\{(1-i)/\sqrt{2}Q_1\}\{\exp(\lambda_5\ell)W_7+i\exp(\lambda_6\ell)W_8\} \tag{38d}$$

The terms with ℓ^2W_3 and ℓ^3W_4 may now be completely disregarded as small terms of order of magnitude Δ^2 in comparison with W_1 and ℓW_2. Thus,

$$\ell^2W_3=\ell^3W_4=0 \tag{39}$$

The number of equations are insufficient to determine the eigenvectors uniquely: two of the eight unknowns are left arbitrary. Additional equations are provided from the condition that the fundamental solutions should be linearly independent of each other. It may be assumed, therefore, that $W_1\neq 0$ and $\ell W_2=0$ for one solution and $W_1=0$ and $\ell W_2\neq 0$ for the other.

For the first solution characterized by $\ell W_2=0$, the following are obtained from Eqs. (38):

$$W_1=\Delta^{-1}(1/M_1)(1+i)\exp(\lambda_5\ell)W_5,\quad \ell W_2=0$$
$$W_7=W_5,\ W_6=W_8=i\exp(\lambda_5\ell)\exp(-\lambda_6\ell)W_5 \tag{40}$$

Substitution of Eqs. (40) into Eq. (33) shows that the terms in the edge-zone solution part in the deflection function are negligible. Thus, one has

$$w=W_0\cos n\theta\sin\omega T \tag{41}$$

where W_0 is an arbitrary constant. Equation (41) represents the Rayleigh type of inextensional mode.

For the second solution characterized by $W_1 = 0$, the following are obtained from Eqs. (38):

$$W_1 = 0,\ \ell W_2 = \Delta^{-1}(1/M_1)(1+i)\exp(\lambda_5\ell)W_5,\ W_7 = -W_5$$
$$W_6 = \{i - \Delta^{1/2}\sqrt{2}(Q_1/M_1)\}\exp(\lambda_5\ell)\exp(-\lambda_6\ell)W_5$$
$$W_8 = -\{i + \Delta^{1/2}\sqrt{2}(Q_1/M_1)\}\exp(\lambda_5\ell)\exp(-\lambda_6\ell)W_5 \qquad (42)$$

Since the terms in the edge-zone solution part are negligible again, the deflection function becomes

$$w = W_0(y/\ell)\cos n\theta \sin\omega T \qquad (43)$$

Equation (43) represents the Love type of inextensional mode.

FR-S3, -S4, -C3, -C4

Let us take, for example, FR-$S4$, assuming FR at $y = \ell$ and $S4$ at $y = -\ell$. The boundary conditions are then given by Eq. (37i) for $y = \ell$ and Eq. (37d) for $y = -\ell$:

$$W_1 = -\ell W_2 + \Delta^{-1}(i/M_1)\{\exp(\lambda_5\ell)W_5 - \exp(\lambda_6\ell)W_6\}$$
$$\ell W_2 = -\Delta^{-3/2}\{(1-i)/\sqrt{2}Q_1\}\{\exp(\lambda_5\ell)W_5 + i\exp(\lambda_6\ell)W_6\}$$
$$W_1 = \ell W_2 = -\Delta^{-2}(S_2/S_1)\ell^3 W_4$$
$$-\Delta^{-3/2}\{(1-i)/\sqrt{2}S_1\}\{\exp(\lambda_5\ell)W_7 + i\exp(\lambda_6\ell)W_8\}$$
$$W_8 = \exp(\lambda_5\ell)\exp(-\lambda_6\ell)W_7 \qquad (44)$$

It follows that

$$W_1 = \ell W_2 = \Delta^{-1}\{(1+i)/2M_1\}\exp(\lambda_5\ell)W_5$$
$$W_6 = \{i - \Delta^{1/2}(Q_1/\sqrt{2}M_1)\}\exp(\lambda_5\ell)\exp(-\lambda_6\ell)W_5\}$$
$$W_8 = \exp(\lambda_5\ell)\exp(-\lambda_6\ell)W_7 = -\Delta^{-1/2}(S_2/\sqrt{2})$$
$$\times \exp(-\lambda_6\ell)\ell^3 W_4$$
$$-\Delta^{1/2}\{(1+i)S_1/2\sqrt{2}M_1\}\exp(\lambda_5\ell)\exp(-\lambda_6\ell)W_5 \qquad (45)$$

Substitution from Eqs. (45) into Eq. (33) shows that the terms in the edge-zone solution part as well as $\ell^2 W_3$ and $\ell^3 W_4$ in the global solution part are negligible in the expression of the deflection function. Thus, it becomes

$$w = W_0(1 + y/\ell)\cos n\theta \sin\omega T \qquad (46)$$

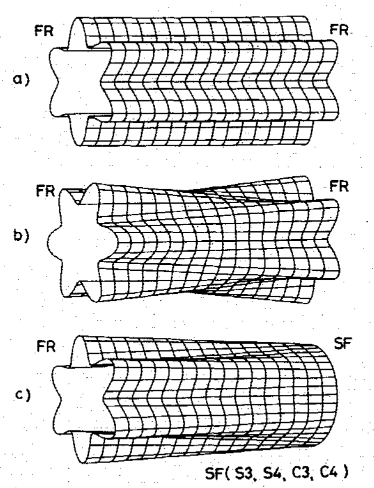

Fig. 1 Schematic of three types of inextensional mode: a) Rayleigh type; b) Love type; c) a linear combination of a) and b).

In a similar manner, the expressions for the deflection function are derived for FR-$S3$, FR-$C3$, and FR-$C4$. It turns out that the deflection function for FR-$S3$ is identical with Eq. (46). As for FR-$C3$ and FR-$C4$, the deflection function becomes identical with Eq. (46) if the following holds:

$$\Delta^{1/2} \ll 1 \qquad (47)$$

Since Eq. (46) is a linear combination of Eqs. (41) and (43), it represents the inextensional vibrations. The three types of inextensional mode are depicted schematically in Fig. 1.

VII. Experimental Verification

An experiment was conducted on a seamless aluminum cylindrical shell having a radius of 33.0 mm and a thickness of 0.155 mm. It has a mass per unit volume of 3.23×10^3 kg/m^3, including the effect of dry powder spray paint, which was painted over the outer surface of the cylinder to achieve a higher optical reflection. Young's modulus of 69.0 GPa and Poisson's ratio of 0.3 were used for the calculations of the frequency parameter ω.

The boundary conditions of FR were established naturally by leaving the end completely free. Those of $S3$, $S4$, $C3$, and $C4$ take $w = N = 0$ in common, so that they may be represented by $w = N = 0$. That is, the end is free in the axial direction, whereas it is restrained in the lateral direction. These boundary conditions will be denoted as $SF(w = N = 0)$. This type of boundary condition may be established by attaching a thin annular plate at an end section. The end plate must have an inplane rigidity that sufficiently resists the lateral deflection of the cylindrical shell, as well as a sufficiently low bending rigidity to permit local displacements of the shell in the axial direction. An annular end plate with an inner radius of 11.0 mm and an outer radius of 33.0 mm was made from an aluminum sheet of a thickness of 0.30 mm, which was attached to an end of the cylindrical shell by adhesive resin. The test specimen with FR-FR ends was screwed to a supporting metal column by a support pin at the center of the cylindrical shell. The test specimen with FR-SF ends was clamped around the inner circle of the annular plate to the supporting column by a pair of nuts. The cross-sectional view of the setups of the test specimens is shown in Fig. 2.

The test specimen was excited by a small piece of thickness-expansion piezoelectric oscillator having a length of 20 mm, a width of 5.0 mm, a thickness of 1.0 mm, and a mass of 0.8 g, including the attached electrical wires. The oscillator was firmly attached by adhesive to the outer surface of the shell parallel lengthwise to the generator. It was located near the supported end in the test specimen with FR-SF ends, or near the support pin in the test specimen with FR-FR ends aligning both the pin and the oscillator along the generator (see Fig. 3). The location of the oscillator was selected in the hope of avoiding or minimizing the effect of the concentrated mass of the oscillator.

Resonant vibrations were detected by holographic interferometry. Change in the real-time fringe pattern was observed as the excitation frequency swept slowly upward. When a rapid

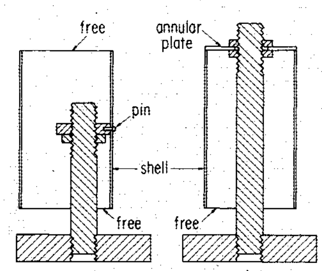

Fig. 2 Cross-sectional view of test specimens: a) *FR-FR*; b) *FR-SF*.

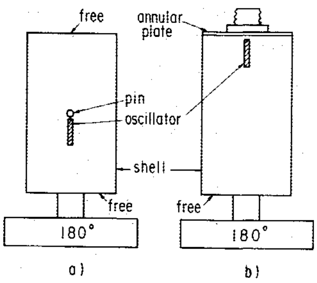

Fig. 3 Location of the oscillator: a) *FR-FR*; b) *FR-SF*.

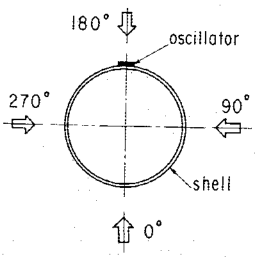

Fig. 4 Direction of hologram recording and observation.

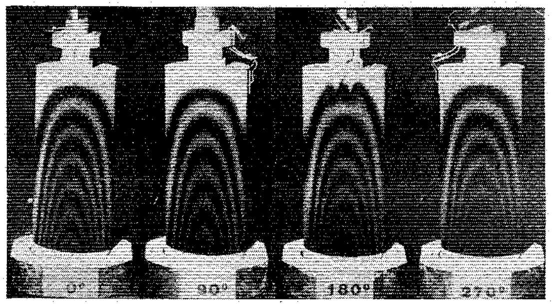

Fig. 5 Sequence of fringe patterns in resonance observed at 0, 90, 180, and 270 deg (*FR-SF*, 94.46 Hz, $n = 2$).

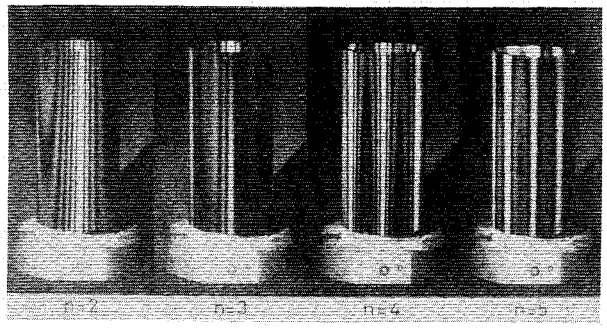

Fig. 6 Fringe patterns of resonant mode of the Rayleigh type in *FR-FR* specimen: $2L/R = 4.26$.

increase in the fringe number was observed, a peak of the fringe number was determined by manually tuning the excitation frequency in a fine range. The time-average holograms were taken at that peak, aiming at the test specimen in four directions 90 deg apart (see Fig. 4). If a sequence of the fringe patterns developed from these holograms showed a regular and periodic pattern consisting of clusters of smooth and symmetric dark fringes separated by the brightest straight nodal lines, the excitation frequency at that peak was determined as the natural frequency, and the natural mode was identified by counting the

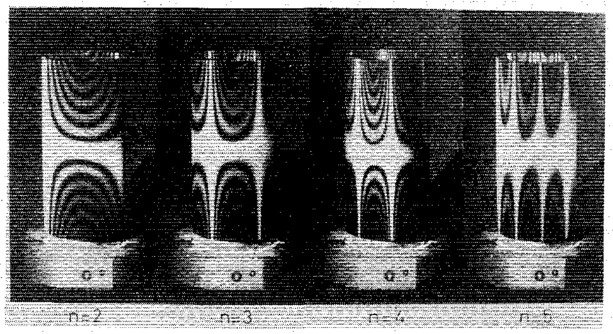

Fig. 7 Fringe patterns of resonant mode of the Love type in *FR-FR* specimen: $2L/R = 4.26$.

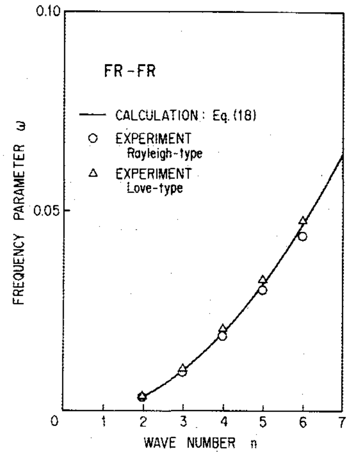

Fig. 8 Natural frequency for the *FR-FR* ends (experiment with $2L/R = 4.26$).

nodal lines. A sequence of the fringe patterns of the test specimen with *FR-SF* ends at an excitation frequency of 94.46 Hz is shown in Fig. 5. Four clusters of evenly distributed fringes are seen spaced equally and periodically around the circumference. The axial nodal lines are invisible in these fringe patterns, but they can be made visible if the position of the hologram recording is rotated by about 45 deg. The vibration in this case can be identified as the natural vibration characterized by $n = 2$ and $f = 94.46$ Hz. It should be noted that the fringes are distorted only slightly in a very narrow region adjacent to the oscillator. The effect of the concentrated mass of the oscillator is therefore insignificant in determining the natural mode. The same holds for other modes of lower order, including those of the test specimen with *FR-FR* ends. In the Rayleigh type of mode, one of the axial nodal lines passed through the support pin. In the Love type of mode, the circumferential nodal line bisecting the axis of the shell passed through the support pin, and two of the axial nodal lines ran parallel to a nearly equidistant from the generator on which the support pin and the oscillator were aligned. In the third type of mode, the supported end formed a circumferential nodal line, and the oscillator was located in the middle of two axial nodal lines.

Representative fringe patterns of the natural modes of the Rayleigh and Love type observed in the test specimen with *FR-FR* ends are shown in Figs. 6 and 7, respectively. The values of ω corresponding to the natural frequencies are plotted against n in Fig. 8. The curve represents the theoretical values of ω_0 calculated from Eq. (18), taking n as a continuous variable. A good agreement is observed between the theoretical and experimental values. It is noted that the natural frequencies of the Love type of mode are always slightly higher than those of the Rayleigh type. Representative fringe patterns observed in the test specimen with the *FR-SF* ends are shown in

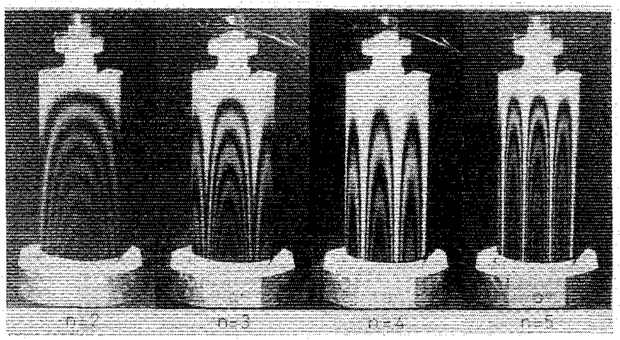

Fig. 9 Fringe patterns of the inextensional resonant mode in *FR-SF* specimen.

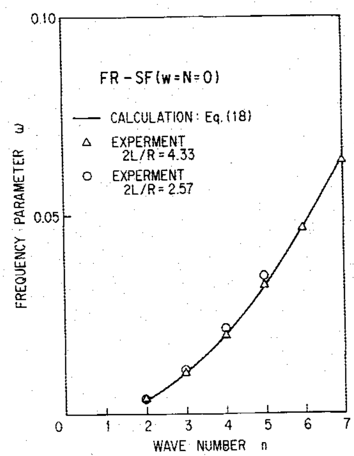

Fig. 10 Natural frequency for the *FR-SF* ends.

Fig. 9. The natural frequencies are plotted against n in Fig. 10. The curve shows again the theoretical values calculated from Eq. (18). The test was conducted on two specimen sizes, one having a length-to-radius ratio of 4.33 and the other of 2.57. The agreement between theory and experiment is excellent. The change in the length of the test specimen does not significantly alter the natural frequencies, as predicted by Eq. (18), which exhibits no dependence on the length parameter. These indicate that the natural vibration of this type is inextensional.

VIII. Conclusions

Asymptotic solutions have been obtained for the eigenvalue problems of the inextensional free vibrations of a circular cylindrical shell. The natural frequency is given by Rayleigh's solution ω_0:

$$\omega_0^2 = \delta n^2(n^2-1)^2/(n^2+1)$$

Nontrivial solutions for the deflection function have been found under five different combinations of the boundary conditions out of the 45 combinations investigated. They are given by

$$w = W_0 \cos n\theta \sin \omega_0 T; \qquad \text{Rayleigh type}$$

$$w = W_0 (y/\ell) \cos n\theta \sin \omega_0 T; \qquad \text{Love type}$$

for *FR-FR*, and

$$w = W_0(1 + y/\ell) \cos n\theta \sin \omega_0 T$$

for *FR-SF* ($S3$, $S4$, $C3$, $C4$). The first two represent the Rayleigh and Love types of inextensional mode. The third is a linear combination of the first two, and thus it also represents an inextensional mode. It immediately occurs to one that the mode shape of the third type is the same as that of a half-length portion of the Love type. This is precisely so in the case of *FR-S3*. In fact, substituting w of the Love type into the right-hand members of Eqs. (15) and solving them for the quantities on the left-hand side, one can prove that $S3$ holds at the mid-length section; $y = 0$. The present analysis has shown that the same holds approximately for $S4$, $C3$, and $C4$.

It has been taken for granted that the inextensional vibrations occur in mode either of the Rayleigh or the Love type when both ends are free or when the shell is indefinitely long. The present analysis has shown that these are not the only possibilities. The inextensional vibrations can also occur for a finite shell when one end is free and the other is supported in such a manner that it can move freely in the axial direction. The existence of these three types of inextensional mode has been proved by an experiment.

The nontrivial solutions for the deflection function have been derived by expanding the exponential functions into power series under the assumption $1/n\ell = O(1)$. The power series expansion is possible even if the assumption is made such that $\Delta n\ell = O(\Delta^{1/2})$. The resulting deflection functions under this assumption are the same as those given if errors of order of magnitude $\Delta^{1/2}$ are admitted. The conclusions of the present analysis therefore apply for cases where the length parameter ℓ is bounded by

$$1/n\ell = O(1) \text{ and } \Delta n\ell = O(\Delta^{1/2})$$

We must now anticipate errors of order of magnitude $\Delta^{1/2}$ to be present in the linear expressions of the deflection function. It should be noted that the same expressions can be obtained for the deflection function if we restrict our attention to a central portion of the shell characterized by $\Delta^2(n\ell)^2(y/\ell)^2 \ll 1$. In this case, however, the error estimation just mentioned is no longer valid uniformly over the entire shell length when ℓ exceeds the limit as specified.

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